МАТЕМАТИКА

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ВЛИЯНИЕ S-C-ПЕРЕСТАНОВОЧНО ПОГРУЖЕННЫХ ПОДГРУПП НА СТРОЕНИЕ КОНЕЧНОЙ ГРУППЫ

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THE INFLUENCE OF S-C-PERMUTABLY EMBEDDED SUBGROUPS ON THE STRUCTURE OF FINITE GROUPS

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Подгруппа H группы G называется s-c-перестановочно погруженной в G, если каждая силовская подгруппа из H является s-условно перестановочной подгруппой в G. В данной работе получены некоторые новые характеризации p-сверхразрешимости или p-нильпотентности для конечных групп при условии, что некоторые из её максимальных или 2-максимальных подгрупп силовских подгрупп являются s-c-перестановочно погруженными. Также в данной работе обобщен ряд известных результатов.

Ключевые слова: *конечная группа, s-c-перестановочно погруженные подгруппы, 2-максимальные подгруппы, силовская подгруппа, p-сверхразрешимая подгруппа, p-нильпотентная подгруппа.*

A subgroup H of a group G is said to be s-c-permutably embedded in G if every Sylow subgroup of H is a Sylow subgroup of some s-conditionally permutable subgroup of G. In this paper, some new characterizations for a finite group to be p-supersoluble or p-nilpotent are obtained under the assumption that some of its maximal subgroups or 2-maximal subgroups of Sylow subgroups are s-c-permutably embedded. A series of known results are generalized.

Keywords: *finite group, s-c-permutably embedded subgroups, 2-maximal subgroups, Sylow subgroup, p-supersoluble group, p-nilpotent group.*

Introduction

Throughout this paper, all groups considered are finite and G denotes a finite group. The terminology and notation are standard, as in [1] and [2].

Let A and B be subgroups of G. A is said to be permutable with B if $AB = BA$. If A is permutable with all subgroups of G, then A is said to be a permutable subgroup [1] (or quasinormal subgroup [3]) of G. The permutable subgroups have many interesting properties. For example, Ore [3] proved that every permutable subgroup of a finite group is subnormal. Itô and Szép [4] proved that for every permutable subgroup H of a finite group G, H/H_G is nilpotent.

However, in general, two subgroups H and T of G may not be permutable in G but G may contain an element x such that $HT^x = T^xH$. Based on the observations, Guo, Shum and Skiba introduced the concept of conditionally permutable subgroup (in more general, the concept of X-permutable subgroup) [5]–[7]: let X be a non-empty subset of G. Then a subgroup A of G is said to be conditionally permutable (X-permutable) in G if for every subgroup T of G, there exists some $x \in G$ ($x \in X$ respectively) such that $AT^x = T^xA$. By using the conditionally permutable subgroups and X-permutable subgroups, many authorse have obtained some new elegant results on the structure of groups (cf. [5]–[8]).

By considering some local conditionally permutable subgroups, Huang and Guo [9] introduced the concept of s-conditionally permutable subgroup: a subgroup H of G is said to be s-conditionally permutable in G if, for every Sylow subgroup T of G, there exists some $x \in G$ such that $HT^x = T^xH$. By Sylow's theorem, we see that a subgroup H of G is s-conditionally permutable in G if and only if for every $p \in \pi(G)$, there exists a Sylow p-subgroup T such that $HT = TH$. As a development of s-conditionally permutable subgroups, Chen and Guo [10] introduced the concept of s-c-permutably embedded subgroups:

Definition **0.1** ([10, Definition 1.1]). *A subgroup H of G is said to be* s-c*-permutably embedded in G if every Sylow subgroup of H is a Sylow subgroup of some s-conditionally permutable subgroup of G.*

Clearly, all permutable subgroups, s-permutable subgroups and s-conditionally permutable subgroups are s-c-permutably embedded. But the converse is not true in general (see, for example, Example 1–2 in [10]).

***Definition* 0.2** ([11], Definition 1.2). *Let d be the smallest generator number of a p-group P and $M_d(P)=\{P_1,\cdots,P_d\}$ be a set of maximal subgroups of P such that*

$$\bigcap_{i=1}^{d} P_i=\Phi(P).$$

Such subset $M_d(P)$ is not unique for a fixed P in general. Let $M(P)$ denotes the family of all maximal subgroups of P. We know that $|M(P)|=(p^d-1)/p-1$, $|M_d(P)|=d$ and $\lim_{n\to\infty}[(p^d-1)/p-1]/d=\infty$, so $|M(P)|\gg|M_d(P)|$.

The purpose of this paper is to go further into the influence of s-c-permutably embedded subgroups on the structure of finite groups. Some new results are obtained and a series of known results are generalized.

1 Preliminaries

Recall that a class $\mathfrak{F}$ of groups is called a formation if $\mathfrak{F}$ is closed under taking homorphic images and subdirect products. A formation $\mathfrak{F}$ is said to be saturated if it contains every group G with $G/\Phi(G)\in\mathfrak{F}$. It is well known that the class of all supersoluble groups is a saturated formation.

For the reader's convenience, we cite some results which are useful in the sequel.

***Lemma* 1.1** [10, Lemma 2.2]. *Suppose that G is a group, $K\lhd G$ and $H\le G$. Then:*

1) If H is s-c-permutably embedded in G, then HK/K is s-c-permutably embedded in G/K.

2) If $K\le H$ and H/K is s-c-permutably embedded in G/K, then H is s-c-permutably embedded in G.

3) If HK/K is s-c-permutably embedded in G/K and $(|H|,|K|)=1$, then H is s-c-permutably embedded in G.

4) If H is s-c-permutably embedded in G, then $H\cap K$ is s-c-permutably embedded in K.

***Lemma* 1.2** [12, Lemma 2.3]. *Let $\mathfrak{F}$ be a saturated formation containing all supersoluble groups and G a group with a normal subgroup N such that $G/N\in\mathfrak{F}$. If N is cyclic, then $G\in\mathfrak{F}$.*

***Lemma* 1.3** [13, Theorem IV 4.7]. *If P is a Sylow p-subgroup of a group G for some $p\in\pi(G)$ and $N\lhd G$ such that $P\cap N\le\Phi(P)$, then N is p-nilpotent.*

***Lemma* 1.4** [14, Lemma II 7.9]. *Let N be a nilpotent normal subgroup of a group G. If $N\cap\Phi(G)=1$, then N is a direct product of some minimal normal subgroups of G.*

***Lemma* 1.5** [15]. *Let p_1 be the minimal prime dividing $|G|$ and p_s the maximal prime dividing $|G|$. If G possesses two supersoluble subgroups H and K with $|G:H|=p_1$ and $|G:K|=p_s$, then G is supersoluble.*

***Lemma* 1.6** [2, Theorem 1.8.6]. *Let H, T be subgroups of G, $N_{G/O_{p'}(G)}(PO_{p'}(G)/O_{p'}(G))$ and $T\subseteq O_p(G)$, then*

$$HT/T\subseteq F_p(G/T)$$

if and only if $H\subseteq F_p(G)$.

***Lemma* 1.7** [16]. *Let p be the minimal prime dividing $|G|$. Suppose G is A_4-free and L is a normal subgroup of G. If G/L is p-nilpotent and $p^3\nmid|L|$, then G is p-nilpotent.*

2 Main results

***Theorem* 2.1.** *Let G be a p-soluble group and P a Sylow p-subgroup of G. Suppose that every member of some fixed $M_d(P)$ is s-c-permutably embedded in G, then G is p-supersoluble.*

*Proof***.** Suppose that the Theorem is false and let G be a counterexample of minimal order. We proceed the proof as follows:

(1) $O_{p'}(G)=1$ and $\Phi(O_p(G))=1$.

Assume that $O_{p'}(G)\ne1$. Then $PO_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of $G/O_{p'}(G)$ and $G/O_{p'}(G)$ is p-soluble. Since

$$|PO_{p'}(G)/O_{p'}(G):P_1O_{p'}(G)/O_{p'}(G)|=$$
$$=|PO_{p'}(G):P_1O_{p'}(G)|=p,$$

$P_1O_{p'}(G)/O_{p'}(G)$ is a maximal subgroup of $PO_{p'}(G)/O_{p'}(G)$. Let

$$P_1O_{p'}(G)/O_{p'}(G)\in M_d(PO_{p'}(G)/O_{p'}(G)),$$

and there must be a subgroup $P_1\in M_d(P)$ such that it holds. Since P_1 is s-c-permutably embedded in G, by Lemma 1.1, $P_1O_{p'}(G)/O_{p'}(G)$ is s-c-permutably embedded in $G/O_{p'}(G)$. Thus, the hypothesis holds for $G/O_{p'}(G)$. By the choice of G, $G/O_{p'}(G)$ is p-supersoluble. It follows that G is p-supersoluble, a contradiction.

Now assume that $\Phi(O_p(G))\ne1$. By the same way, we see that the hypothesis holds for $G/\Phi(O_p(G))$. The minimal choice of G implies that $G/\Phi(O_p(G))$ is p-supersoluble. Since the class of all p-supersoluble groups is a saturated formation, we obtain that G is p-supersoluble, a contradiction.

(2) Every minimal normal subgroup of G contained in $O_p(G)$ is of order p and $O_p(G)\cap\Phi(G)=1$.

Since G is p-soluble and $O_{p'}(G)=1$, we have $O_p(G)\neq 1$. Let N be an arbitrary minimal normal subgroup of G contained in $O_p(G)$. By Lemma 1.1, we see that the quotient group G/N satisfies the hypothesis. The minimal choice of G implies that G/N is p-supersoluble. If $N\leq \Phi(P)$ then G is p-supersouble, a contradiction. Thus $N\nleq \Phi(P)$. Since $\Phi(P)=\bigcap_{i=1}^{d} P_i$, where $P_i\in M_d(P)$, we may without loss of generality assume that $N\nleq P_1$. Let $N_1=N\cap P_1$. Then

$|N:N_1|=|N:N\cap P_1|=|NP_1:P_1|=|P:P_1|=p$.

Hence, N_1 is a maximal subgroup of N. Since P_1 is s-c-permutably embedded in G, there exists an s-conditionally permutable subgroup A of G such that P_1 is a Sylow p-subgroup of A. Then for an arbitrary prime divisor q of $|G|$ with $p\neq q$, there exits a Sylow q-subgroup Q of G such that $AQ=QA$. Since N_1 is a maximal subgroup of N and $N_1=N\cap P_1\leq N\cap A\leq N\cap AQ\leq N$, we have that $N_1=N\cap AQ$ or $N=N\cap AQ$. If $N=N\cap AQ$, then $N\leq AQ$ and so $P=P_1N\leq P_1AQ=AQ$. This implies that $P=P_1$, which is impossible. Hence $N_1=N\cap AQ$. It follows that $N_1\lhd AQ$. consequently $Q\leq N_G(N_1)$. On the other hand, since $N_1\lhd N$ and $N_1\lhd P_1$, $N_1\lhd P_1N=P$. This implies that $N_1\lhd G$. But since N is a minimal normal subgroup of G, $N_1=1$ and N is a cyclic subgroup of order p. It follows that $N\cap P_1=1$. By Huppert [13, Theorem I. 17. 4], there exists a subgroup M of G such that $G=NM$ and $N\cap M=1$. Hence $N\nleq\Phi(G)$. It follows that $O_p(G)\cap\Phi(G)=1$.

(3) $O_p(G)=R_1\times\cdots\times R_r$, where R_i (i=1,…,r) is a minimal normal subgroup of G of order p.

It follows directly from (2) and Lemma 1.4.

(4) The final contradiction.

Since $G/C_G(R_i)$ is isomorphic with some subgroup of $Aut(R_i)$ which is a cyclic group,

$$G/C_G(O_p(G))=G/(\bigcap_{i=1}^{r} C_G(R_i))$$

is p-supersoluble. On the other hand, since G is p-soluble and $O_{p'}(G)=1$, $C_G(O_p(G))\leq O_p(G)$ by [2, Theorem 1.8.19]. Thus $G/O_p(G)$ is p-supersoluble. It follows from (3) that G is p-supersoluble. The final contradiction completes the proof.

As immediate corollaries of Theorem 2.1, we have the following:

***Corollary* 2.1.1.** *Let G be a p-soluble group and P a Sylow p-subgroup of G. Suppose that every member of some fixed $M_d(P)$ is s-conditionally permutable in G, then G is p-supersoluble.*

***Corollary* 2.1.2.** *Let G be a soluble group. If every member of some fixed $M_d(P)$ is s-conditionally permutable in G, for each prime p in $\pi(G)$ and a Sylow p-subgroup P of G, then G is supersoluble.*

***Corollary* 2.1.3.** [9, Lemma 4.1]. *Let G be a p-soluble group. If every maximal subgroup of every Sylow p-subgroup of G is s-conditionally permutable in G, then G is p-supersoluble.*

***Corollary* 2.1.4.** [11, Theorem 1.3]. *Let G be a p-soluble group and P a Sylow p-subgroup of G. Suppose that every member of some fixed $M_d(P)$ is SS-quasinormal in G, then G is p-supersoluble.*

Following [17], a subgroup H of a group G is said to be s-semipermutable in G if for every prime p with $(p,|H|)=1$, H permutes with every Sylow p-subgroup of G.

***Corollary* 2.1.5.** *Let G be a p-soluble group and P a Sylow p-subgroup of G. Suppose that every member of some fixed $M_d(P)$ is s-semipermutable in G, then G is p-supersoluble.*

***Theorem* 2.2.** *Let G be a p-soluble group and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every member of some fixed $M_d(P)$ is s-c-permutably embedded in G, then G is p-nilpotent.*

*Proof***.** Suppose that the theorem is false and let G be a counterexample of minimal order. Then:

(1) $O_{p'}(G)=1$.

Assume that $O_{p'}(G)\neq 1$. Then $PO_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of $G/O_{p'}(G)$ and by [2, Lemma 3.6.10]

$$N_{G/O_{p'}(G)}(PO_{p'}(G)/O_{p'}(G))=N_G(P)O_{p'}(G)/O_{p'}(G)$$

is p-nilpotent. Since

$$|PO_{p'}(G)/O_{p'}(G):P_1O_{p'}(G)/O_{p'}(G)|=$$
$$=|PO_{p'}(G):P_1O_{p'}(G)|=p,$$

$P_1O_{p'}(G)/O_{p'}(G)$ is a maximal subgroup of $PO_{p'}(G)/O_{p'}(G)$, so

$$P_1O_{p'}(G)/O_{p'}(G)\in M_d(PO_{p'}(G)/O_{p'}(G)),$$

and there must be a subgroup $P_1\in M_d(P)$ such that it holds. By the hypothesis, P_1 is s-c-permutably embedded in G. Then by Lemma 1.1, we see that

$P_1O_{p'}(G)/O_{p'}(G)$ is s-c-permutably embedded in $G/O_{p'}(G)$. Thus the hypothesis holds for $G/O_{p'}(G)$. The minimal choice of G implies that $G/O_{p'}(G)$ is p-nilpotent and consequently G is p-nilpotent, a contradiction.

(2) $O_p(G) = R_1 \times \cdots \times R_r$, where R_i (i=1,…,r) is a minimal normal subgroup of G of order p (see the proof (3) of Theorem 2.1).

(3) The final contradiction.

Since $G/C_G(R_i)$ is an abelian group of exponent $p-1$, $P \le \bigcap_{i=1}^{d} C_G(R_i) = C_G(O_p(G))$ by (2). Moreover, by (1) and [2, Theorem 1.8.18], $C_G(O_p(G)) \le O_p(G)$. Hence $P = O_p(G)$ and therefore $G = N_G(P)$ is p-nilpotent. The final contradiction completes the proof.

***Corollary* 2.2.1.** *Let G be a p-soluble group and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every member of some fixed $M_d(P)$ is s-conditionally permutable in G, then G is p-nilpotent.*

***Corollary* 2.2.2.** *Let p be a prime dividing the order of G and H a p-soluble normal subgroup of G such that G/H is p-nilpotent. Suppose that P is a Sylow p-subgroup of H. If $N_G(P)$ is p-nilpotent and every member in $M_d(P)$ is s-conditionally permutable in G, then G is p-nilpotent.*

Proof. Since $N_H(P) \le N_G(P)$, $N_H(P)$ is p-nilpotent. By Lemma 1.1(1), every member in $M_d(P)$ is s-c-permutably embedded in H. Hence by Theorem 2.2, H is p-nilpotent. Let N be the normal Hall p'-subgroup of H. Then $N \lhd G$. We claim that G/N (with respect to H/N) satisfies the hypothesis of the corollary. In fact, $H/N \lhd G/N$, $(G/N)/(H/N) \cong G/H$ is p-nilpotent and $N_{G/N}(NP/N) = N_G(P)N/N$ is p-nilpotent. Let $\overline{P_1}$ be a maximal subgroup of $\overline{P}$, where $\overline{P_1} \in M_d(\overline{P})$. Obviously, there exists a $P_1 \in M_d(P)$ such that $P_1N/N = \overline{P}$. Since P_1 is s-c-permutably embedded in G, P_1N/N is s-c-permutably embedded in G/N by Lemma 1.1. Hence our claim holds. If $N \ne 1$, then G/N is p-nilpotent by induction. It follows that G is p-nilpotent. If $N = 1$, then $H = P$ is a p-group. In this case, $G = N_G(P)$ is p-nilpotent. This completes the proof.

***Theorem* 2.3.** *Let $\mathfrak{F}$ be a saturated formation containing the class $\mathfrak{U}$ of all supersoluble groups. A group $G \in \mathfrak{F}$ if and only if there exists a soluble normal subgroup H of G such that $G/H \in \mathfrak{F}$ and for every Sylow subgroup P of H, every member of $M_d(P)$ is s-c-permutably embedded in G.*

Proof. The necessary is obvious. We only need to prove the sufficiency. Suppose that it is false and let G be a counterexample of minimal order. Let q be the largest prime divisor of $|H|$ and Q be a sylow q-subgroup of H. Then:

(1) $Q \lhd G$.

By Lemma 1.1(1), every member of $M_d(P)$ is s-c-permutably embedded in H. Hence by Theorem 2.1, H is supersoluble. It follows that $Q \lhd G$.

(2) Q is a Sylow q-subgroup of G.

Suppose that Q is not a Sylow q-subgroup of G. Let p be the smallest prime dividing $|G/Q|$ and r the largest prime dividing $|G/Q|$. Furthermore, $(G/Q)/(H/Q) = G/H$ is supersoluble and Lemma 1.1 shows that G/Q satisfies the condition of the Theorem and by the choice of G, G/Q is supersoluble. So G/Q contains two subgroups M_1/Q and M_2/Q with $|G : M_1| = p$ and $|G : M_2| = r$. By Lemma 1.1, $(M_i, Q)(i = 1,2)$ satisfies the condition. By the choice of G, $M_i (i = 1,2)$ is supersoluble. Now by Lemma 1.5, G is supersoluble, a contradiction. Then (2) holds.

(3) Every minimal normal subgroup of G contained in Q is of order q.

Let N be an arbitrary minimal normal subgroup of G contained in Q. Since $N \not\le \Phi(Q)$, we can, without loss of generality, assume that $N \not\le Q_1$, where $Q_1 \in M_d(P)$. Let $N_1 = N \cap Q_1$. Then

$$|N : N_1| = |N : N \cap Q_1| = |NQ_1 : Q_1| = |Q : Q_1| = q.$$

Hence N_1 is the maximal subgroup of N and so $N_1 \lhd N$. Since Q_1 is s-c-permutably embedded in G, there exists an s-conditionally permutable subgroup A of G such that Q_1 is a Sylow q-subgroup of A. This means that for an arbitrary prime divisor p of $|G|$ with $q \ne p$, there exits a Sylow p-subgroup P of G such that $AP = PA$. Since N_1 is the maximal subgroup of N and $N_1 = N \cap Q_1 \le N \cap A \le N \cap AP \le N$, $N_1 = A \cap NP$ or $N = N \cap AP$. If $N = N \cap AP$, then $N \le AP$ and hence $Q = Q_1N \le Q_1AP = AP$. This implies $Q = Q_1$, which is impossible. This shows that $N_1 = N \cap AP \lhd AP$ and therefore $A \le N_G(N_1)$, so $Q = Q_1N \le N_G(N_1)$. Since Q is a Sylow q-subgroup of G. then $N_1 \lhd G$ and so $N_1 = 1$. Hence N is a cyclic subgroup of order q.

(4) $Q \cap \Phi(G) = 1$.

Assume $Q \cap \Phi(G) \neq 1$ and let N be a minimal normal subgroup of G contained in $Q \cap \Phi(G)$. Then, clearly, G/N (with respect to H/N) satisfies the hypothesis. Hence $G/N \in \mathfrak{F}$ by the choice of G. It follows from (3) and Lemma 1.2 that $G \in \mathfrak{F}$ a contradiction.

(5) $Q = R_1 \times \cdots \times R_r$, where R_i $(i = 1, \cdots, r)$ is a minimal normal subgroup of G of order p.

It follows directly from Lemma 1.4.

(6) The final contradiction.

It is easy to see that G/Q satisfies the hypotheses. Hence $G/Q \in \mathfrak{F}$. Since every chief factor of G contained in Q is a cyclic group of order q. By Lemma 1.2, we obtain that $G \in \mathfrak{F}$. The final contradiction completes the proof.

***Corollary* 2.3.1.** *Let $\mathfrak{F}$ be a saturated formation containing the class $\mathfrak{U}$ of all supersoluble groups. A group $G \in \mathfrak{F}$ if and only if there exists a soluble normal Hall subgroup H of G such that $G/H \in \mathfrak{F}$ and for every Sylow subgroup P of H, every member of $M_d(P)$ is s-conditionally permutable in G.*

***Theorem* 2.4.** *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersoluble groups. A group $G \in \mathfrak{F}$ if and only if there exists a soluble normal subgroup H of G such that $G/H \in \mathfrak{F}$ and, for every Sylow p-subgroup P of $F(H)$ satisfying $(|G : F(H)|, p) = 1$, every member of $M_d(P)$ is s-c-permutably embedded in G.*

*Proof***.** The necessary is obvious. We only need to prove the sufficiency. Suppose that the assertion is not true and let G be a counterexample of minimal order. Let P be an arbitrary Sylow p-subgroup of $F(H)$. Then P char $F(H) \lhd G$ and so $P \lhd G$. Since $\Phi(P)$ char $P \lhd G$, $\Phi(P) \lhd G$. We now proceed the proof as follows:

(1) $\Phi(P) = 1$.

Assume that $\Phi(P) \neq 1$. Obviously,

$$(G/\Phi(P))/(H/\Phi(P)) \cong G/H \in \mathfrak{F}.$$

Let $F(H/\Phi(P)) = T/\Phi(P)$, then $F(H) \subseteq T$. On the other hand, since $\Phi(P) \subseteq \Phi(G)$, T is nilpotent by [18, Theorem IV 3.7]. It follows that $T \subseteq F(H)$ and so $T = F(H)$. Since $\Phi(P) = \bigcap_{i=1}^{d} P_i$, where $P_1 \in M_d(P)$, $P_1/\Phi(P)$ is a maximal subgroup of $P/\Phi(P)$. Obviously, $M_d(P/\Phi(P)) = \{P_1/\Phi(P), \cdots, P_d/\Phi(P)\}$ and $(|G/\Phi(P) : F(H/\Phi(P))|, p) = (|G : F(H)|, p) = 1$. Since P_1 is s-c-permutably embedded in G by hypotheses, by Lemma 1.1, $P_1/\Phi(P)$ is s-c permutably embedded in $G/\Phi(P)$. Let $Q_1\Phi(P)/\Phi(P)$ be a maximal subgroup of Sylow q-subgroup $Q\Phi(P)/\Phi(P)$ of $F(H)/\Phi(P) = F(H/\Phi(P))$, where $q \neq p$, Q is a Sylow q-subgroup of $F(H)$ and $Q_1 \in M_d(Q)$. By the hypothesis, Q_1 is s-c-permutably embedded in G. Hence by Lemma 1.1, $Q_1\Phi(P)/\Phi(P)$ is s-c-permutably embedded in $G/\Phi(P)$. This shows that $G/\Phi(P)$ with respect to $H/\Phi(P)$ satisfies the hypothesis. The minimal choice of G implies that $G/\Phi(P) \in \mathfrak{F}$. Then, since $\mathfrak{F}$ is a saturated formation, we obtain that $G \in \mathfrak{F}$, a contradiction.

(2) Every minimal normal subgroup of G contained in P is of order p.

Let N be an arbitrary minimal normal subgroup of G contained in P. Since $\Phi(P) = 1$, $N \nleq \Phi(P)$. Without loss of generality, we may assume that $N \nleq P_1$, where $P_1 \in M_d(P)$. Let $N_1 = N \cap P_1$. Since $|N : N_1| = p$, N_1 is a maximal subgroup of N and so $N_1 \lhd N$. Since P_1 is s-c-permutably embedded in G, there exits an s-conditionally permutable subgroup A of G such that P_1 is a Sylow p-subgroup of A. This means that for an arbitrary prime divisor q of $|G|$ with $p \neq q$, there exits a Sylow q-subgroup Q of G such that $AQ = QA$. Since N_1 is a maximal subgroup of N and $N_1 = N \cap P_1 \leq N \cap AQ \leq N$, $N_1 = A \cap NQ$ or $N = N \cap AQ$. If $N = N \cap AQ$, then $N \leq AQ$ and hence $P = P_1N \leq P_1AQ = AQ$. This implies $P = P_1$, which is impossible. Hence we may assume that $N_1 = N \cap AQ$. Because $N \lhd G$, $N_1 \lhd AQ$. It follows that $A \leq N_G(N_1)$ and so $P = P_1N \lhd N_G(N_1)$. Since $(|G : F(H)|, p) = 1$, P is also a Sylow p-subgroup of G. This shows that $N_1 \lhd G$. Since N is a minimal normal subgroup of G, $N_1 = 1$ and thus N is a cyclic subgroup of order p.

(3) The final contradiction.

By (2), we know that $F(H) = R_1 \times \cdots \times R_s$, where R_i $(i = 1, \cdots, s)$ is a minimal normal subgroup of G of order p. Since $G/C_G(R_i) \simeq Aut(R_i)$, $G/C_G(R_i)$ is cyclic. Therefore, $G/(\bigcap_{i=1}^{s} C_G(R_i)) = G/(C_G(F(H)) \in \mathfrak{F}$, we have $G/C_G(F(H)) \in \mathfrak{F}$. Therefore, $G/C_H(F(H)) = G/(H \cap C_G(F(H))) \in \mathfrak{F}$. Since $F(H)$ is an abelian group, $F(H) \subseteq C_H(F(H))$. On the other hand, we have $C_H(F(H)) \subseteq F(H)$ for H is soluble. Hence, $F(H) = C_H(F(H))$. This implies that $G/F(H) \in \mathfrak{F}$.

Consequently $G \in \mathfrak{F}$. The final contradiction completes the proof.

***Corollary* 2.4.1.** *Let $\mathfrak{F}$ be a saturated formation containing the class of all supersoluble groups. A group $G \in \mathfrak{F}$ if and only if there exists a soluble normal subgroup H of G such that $G/H \in \mathfrak{F}$ and, for every Sylow p-subgroup P of $F(H)$ satisfying $(|G:F(H)|, p) = 1$, every member of $M_d(P)$ is s-conditionally permutable in G.*

Recall that a subgroup H of G is said to be a 2-maximal subgroup of G if H is a maximal subgroup of some maximal subgroup of G. A group G is A_4-free if there are no subgroups in G for which A_4 is an isomorphic image.

***Theorem* 2.5.** *Let G be an A_4-free p-soluble group and p the minimal prime dividing $|G|$. Then G is p-nilpotent if and only if there exists a normal subgroup H of G such that G/H is p-nipotent and every 2-maximal subgroup of all Sylow p-subgroups of H is s-c-permutably embedded in G.*

Proof. The necessity is obvious. We only need to prove the sufficiency. Suppose that the assertion is false and let G be a counterexample of minimal order. Then

(1) $O_{p'}(G) = 1$.

Suppose that $O_{p'}(G) \neq 1$. Obviously,

$$(G/O_{p'}(G))/(HO_{p'}(G)/O_{p'}(G)) \cong$$
$$\cong (G/H)/(HO_{p'}(G)/H)$$

is p-nilpotent. Let $R/O_{p'}(G)$ be a Sylow p-subgroup of $HO_{p'}(G)/O_{p'}(G)$ and $P/O_{p'}(G)$ a 2-maximal subgroup of $R/O_{p'}(G)$. Then there must be a 2-maximal subgroup P_1 of some sylow p-subgroup of H such that $P = P_1O_{p'}(G)$. By Lemma 1.1, every 2-maximal subgroup of $R/O_{p'}(G)$ is s-c-permutably embedded in $G/O_{p'}(G)$. Thus the hypothesis holds for $G/O_{p'}(G)$. By the choice of G, $G/O_{p'}(G)$ is p-nilpotent. It follows that G is p-nilpotent, a contradiction.

(2) There exists an unique minimal normal subgroup L of G and $L = C_G(L) = O_p(G)$.

Let L be an arbitrary minimal normal subgroup of G. If $L \subseteq H$, then by Lemma 1.1, G/L satisfies the hypothesis. If $L \nsubseteq H$, then $H \cap L = 1$. Let φ be an isomorphism between HL/L and H such that $\varphi(hL) = h$. Suppose that U/L is an arbitrary 2-maximal subgroup of a Sylow p-subgroup of HL/L, then $V = \varphi(U/L)$ is a 2-maximal subgroup of a Sylow p-subgroup of H and $U = VL$. By Lemma 1.1, U/L is s-c-permutably embedded in G/L. Hence the hypothesis holds for G/L. Since the class of all p-nilpotent groups is a saturated formation, L is the unique minimal normal subgroup of G and $L \nsubseteq \Phi(G)$. By (1) and the hypothesis, L is a p-group and so $L = C_G(L) = O_p(G)$. Thus (2) holds.

(3) $G = L \rtimes M$, where $p^3 \| |L|$ and M is p-nilpotent.

Since $L \nleq \Phi(G)$, there exists a maximal subgroup M of G such that $G = LM$. Since L is an elementary abelian p-group , so $G = L \rtimes M$ and $M \cong G/L$ is p-nilpotent. It is easy to see that G is p-nilpotent by Lemma 1.7 if $p^3 \nmid |L|$, a contradiction.

(4) The final contradiction.

Suppose that G_p is a Sylow p-subgroup of G. Then there exists a Sylow p-subgroup M_p of M such that $G_p = LM_p$. Since $|L| \geqslant p^3$, there exists a 2-maximal subgroup P_1 of G_p such that $M_p \leq P_1$. Let $P_2 = P_1 \cap H$. Obviously, $G_p \cap H = H_p$ is a Sylow p-subgroup of H and $P_2 = P_1 \cap H = P_1 \cap H_p$. Since $G_p = LM_p = LP_1 = H_pP_1$, $|H_p : P_2| = |H_pP_1 : P_1| = p^2$ and so P_2 is a 2-maximal subgroup of H_p. Hence by hypothesis, P_2 is s-c-permutably embedded in G. So there exists an s-conditionally permutable subgroup A of G such that P_2 is a Sylow p-subgroup of A. Then for an arbitrary prime divisor q of $|G|$ with $q \neq p$, there exists a Sylow q-subgroup G_q of G such that $AG_q = G_qA$. Let $L_1 = L \cap P_2$. Then

$$|L : L_1| = |L : L \cap P_2| = |LP_2 : P_2| =$$
$$= |L(P_1 \cap H) : P_2| = |LP_1 \cap H : P_2| = |H_p : P_2| = p^2,$$

which means that L_1 is a 2-maximal subgroup of L. Since $L_1 = L \cap P_2 = L \cap A = L \cap AG_q$, $L_1 \lhd AG_q$. It follows that $G_q \subseteq N_G(L_1)$. On the other hand, since $L \cap P_2 = L \cap H \cap P_1 \lhd P_1$ and $L \cap P_2 \lhd L$, $L_1 \lhd G_p$. Hence $L_1 \lhd P_1L = G$. But since L is the minimal normal subgroup of G, $L_1 = 1$, which contradicts to $p^3 \| |L|$. The final contradiction completes the proof.

***Corollary* 2.5.1.** *Let G be an A_4-free p-soluble group and p the minimal prime dividing $|G|$. Then G is p-nilpotent if and only if there exists a normal subgroup H of G such that G/H is p-nilpotent and every 2-maximal subgroup of all Sylow p-subgroups of H is s-conditionally permutable in G.*

***Corollary* 2.5.2.** *Let G be an A_4-free soluble group. Then G is a Sylow tower group of supersoluble type if and only if there exists a normal subgroup H of G such that G/H is a Sylow tower group of supersoluble type and every 2-maximal subgroup of all Sylow subgroups of H is s-c-permutably embedded in G.*

Proof. Suppose that p is a minimal prime divisor of $|G|$. By Theorem 2.5, G is p'-closed. Let M be a Hall p'-subgroup of G. Then M is a Sylow tower group of supersoluble type by induction and consequently G is a Sylow tower group of supersoluble type.

***Theorem* 2.6.** *Let G be an A_4-free p-soluble group and p the minimal prime dividing $|G|$. Then G is p-nilpotent if and only if there exists a normal subgroup H of G such that G/H is p-nilpotent and every 2-maximal subgroup of all Sylow p-subgroups of $F_p(H)$ is s-c-permutably embedded in G.*

Proof. The necessity is obvious. We only need to prove the sufficiency. Assume that the assertion is false and let G be a counterexample of minimal order. Then

(1) $O_{p'}(G) = \Phi(G) = 1$.

Let $T \in \{O_{p'}(G), \Phi(G)\}$. Suppose that $T \neq 1$. Obviously,

$$(G/T)/(HT/T) \cong G/HT \cong (G/H)/(HT/H)$$

is p-nilpotent. By Lemma 1.6 and [3, Corollary 1.8.1], we have that $F_p(HT/T) = F_p(H)T/T$. Assume that R/T is a Sylow p-subgroup of $F_p(HT/T)$ and P/T is an arbitrary 2-maximal subgroup of R/T. Then there must be some 2-maximal subgroup P_1 of a Sylow p-subgroup of $F_p(H)$ such that $P = P_1T$. By Lemma 1.1, every 2-maximal subgroup of R/T is s-c-permutably embedded in G/T. Hence the hypothesis holds for G/T. By the choice of G, G/T is p-nilpotent and so G is p-nilpotent, a contradiction. Thus (1) holds.

(2) For an arbitrary minimal normal subgroup L of G, we have that $L \subseteq H$.

Assume that L is an arbitrary minimal normal subgroup of G. If $L \nsubseteq H$, then $F_p(H) \cap L = H \cap L = 1$. Let φ be an isomorphism between HL/L and H such that $\varphi(hL) = h$. Then $\varphi(F_p(HL)/L) = F_p(H) = \varphi(F_p(H)L/L)$. It follows that $F_p(HL)/L = F_p(H)L/L$. Suppose that U/L is an arbitrary 2-maximal subgroup of a Sylow p-subgroup of $F_p(HL)/L = F_p(H)L/L$. Then $V = \varphi(U/L)$ is a 2-maximal subgroup of some Sylow p-subgroup of $F_p(H)$ and $U = VL$. By Lemma 1.1, U/L is s-c-permutably embedded in G/L. Hence the hypothesis holds on G/L. By the choice of G, G/L is p-nilpotent and so $G \cong G/(H \cap L)$ is p-nilpotent, a contradiction.

(3) $F_p(H) = O_p(H) = F(H) =$
$= F(G) = O_p(G) = Soc(G)$.

It is directly obtained from (1) and (2).

(4) Let L be an arbitrary minimal normal subgroup of G. Then $L \subseteq Z(G)$.

In view of (1), there exists a maximal subgroup M of G such that $G = L \rtimes M$. Assume that $p^3 \| |L|$. Let M_p be a Sylow p-subgroup of M and G_p a Sylow p-subgroup of G such that $M_p \subseteq G_p$. Obviously, $|G_p : M_p| = |L| \geqslant p^3$. So there exists a 2-maximal subgroup P_1 of G_p such that $M_p \leq P_1$. By (2), we have that $L \subseteq F(H)$. Let $P_2 = P_1 \cap F(H)$. Clearly, $G_p \cap F(H) = F(H)_p$ is a Sylow p-subgroup of $F(H)$ and

$$P_2 = P_1 \cap F(H) = P_1 \cap F(H)_p.$$

Since $G_p = LM_p = LP_1 = F(H)_p P_1$ and

$$|F(H)_p : P_2| = |F(H)_p P_1 : P_1| = p^2,$$
$$P_2 = P_1 \cap F(H)$$

is a 2-maximal subgroup of $F(H)_p$. By hypothesis, P_2 is s-c-permutably embedded in G. Hence there exists an s-conditionally permutable subgroup A such that P_2 is a Sylow p-subgroup of A. Then for an arbitrary prime divisor q of $|G|$ with $q \neq p$, there exists a Sylow q-subgroup G_q of G such that $AG_q = G_qA$. Let $L_1 = L \cap P_2$. Then

$$|L : L_1| = |L : L \cap P_2| = |LP_2 : P_2| =$$
$$= |L(P_1 \cap F(H)) : P_2| =$$
$$= |LP_1 \cap F(H) : P_2| = |F(H)_p : P_2| = p^2,$$

which implies that L_1 is a 2-maximal subgroup of L. Since $L_1 = L \cap P_2 = L \cap A = L \cap AG_q$, $L_1 \lhd AG_q$. It follows that $G_q \subseteq N_G(L_1)$. On the other hand, since $L \cap P_2 = L \cap H \cap P_1 \lhd P_1$ and $L \cap P_2 \lhd L$, $L_1 \lhd G_p$. Hence $L_1 \lhd P_1L = G$. But since L is a minimal normal subgroup of G, $L_1 = 1$, which contradicts to $p^3 \| |G|$. This contradiction shows that $|L| = p$ or p^2. Let T be a Hall p'-subgroup of G. Then LT is a nilpotent subgroup of G and so $L \subseteq C_G(T)$. On the other hand, since $L \subseteq G_p$, $L \cap Z(G_p) \neq 1$. Hence $|L| = p$ and $L \subseteq Z(G)$.

(4) The final contradiction.

By (3) and (4), we have that $F(G)=Soc(G)\subseteq Z(G)$. It follows that $G=F(G)$ is nilpotent. The final contradiction completes the proof.

Corollary 2.6.1. *Let G be an A_4-free p-soluble group and p the minimal prime dividing $|G|$. Then G is p-nilpotent if and only if there exists a normal subgroup H of G such that G/H is p-nilpotent and every 2-maximal subgroup of all Sylow p-subgroups of $F_p(H)$ is s-conditionally permutable in G.*

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